Analysis 1, Summer 2023

List 9

Integration by parts, area, volume of revolution

220. Fill in the missing parts of the table:

$f =$	$\sin(x)$	$\ln(x)$	x^3	$\boxed{\frac{1}{2}x^2}$	$\boxed{\ln(x)}$	$\boxed{-\cos(x)}$
$\mathrm{d}f =$	$\cos(x)\,\mathrm{d}x$	$\boxed{\frac{1}{x}\,\mathrm{d}x}$	$\boxed{3x^2\,\mathrm{d}x}$	$x\,\mathrm{d}x$	$\frac{\mathrm{d}x}{x}$	$\sin(x)\,\mathrm{d}x$

221. Find the derivative of $2xe^{2x}$.

Integration by parts for indefinite integrals:

$$\int u\,dv = uv - \int v\,\mathrm{d}u.$$

222. Use integration by parts with $u = 4x$ and $\mathrm{d}v = e^{2x}\,\mathrm{d}x$ to evaluate $\int 4xe^{2x}\,\mathrm{d}x$.

$\boxed{(2x-1)e^{2x} + C}$

223. Use integration by parts with $u = \ln(x)$ and $\mathrm{d}v = 1\,\mathrm{d}x$ to find $\int \ln(x)\,\mathrm{d}x$.

$\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$, and $v = x$, so

$$\int \ln(x) 1\,\mathrm{d}x + \int x\frac{1}{x}\,\mathrm{d}x = x\ln(x)$$
$$\int \ln(x)\,\mathrm{d}x + \int 1\,\mathrm{d}x = x\ln(x)$$
$$\int \ln(x)\,\mathrm{d}x + x + C = x\ln(x)$$
$$\int \ln(x)\,\mathrm{d}x = \boxed{x\ln(x) - x + C}$$

224. Find the following indefinite integrals using integration by parts:

(a) $\int x\sin(x)\,\mathrm{d}x$

With $u = x$ and $\mathrm{d}v = \sin(x)\,\mathrm{d}x$, we have $\mathrm{d}u = \mathrm{d}x$ and $v = -\cos(x)$.

$$uv - \int v\,\mathrm{d}u = x(-\cos(x)) + \int \cos(x)\,\mathrm{d}x = \boxed{\sin(x) - x\cos(x) + C}.$$

(b) $\int x\cos(8x)\,\mathrm{d}x = \boxed{\frac{1}{8}x\sin(8x) + \frac{1}{64}\cos(8x) + C}$

(c) $\int \frac{\ln(x)}{x^5}\,\mathrm{d}x = \boxed{-\frac{\ln(x)}{4x^4} - \frac{1}{16x^4} + C}$

(d) $\int x^2 \cos(4x)\,\mathrm{d}x$

With $u = x^2$ and $\mathrm{d}v = \cos(4x)\,\mathrm{d}x$, we have $\mathrm{d}u = 2x\,\mathrm{d}x$ and $v = \frac{1}{4}\sin(4x)$.

$$\int x^2 \cos(4x)\,\mathrm{d}x = uv - \int v\,\mathrm{d}u = \tfrac{1}{4}x^2\sin(4x) - \int \tfrac{1}{4}\sin(4x)2x\,\mathrm{d}x$$
$$= \tfrac{1}{4}x^2\sin(4x) - \tfrac{1}{2}\int x\sin(4x)\,\mathrm{d}x$$

This new integral also requires integration by parts (it is extremely similar to parts a and b).

New $u = x$ and $\mathrm{d}v = \sin(4x)\,\mathrm{d}v$ gives $\mathrm{d}u = \mathrm{d}x$ and $v = \frac{-1}{4}\cos(4x)$, so

$$\int x\sin(4x)\,\mathrm{d}x = x(\tfrac{-1}{4}\cos(4x)) - \int \tfrac{-1}{4}\cos(4x)\,\mathrm{d}x$$
$$= \tfrac{-1}{4}x\cos(4x) - \tfrac{1}{16}\sin(4x) + C$$

and then

$$\int x^2\cos(4x)\,\mathrm{d}x = \tfrac{1}{4}x^2\sin(4x) - \tfrac{1}{2}\left(\tfrac{-1}{4}x\cos(4x) - \tfrac{1}{16}\sin(4x) + C\right)$$
$$= \tfrac{1}{4}x^2\sin(4x) + \tfrac{1}{8}x\cos(4x) - \tfrac{1}{32}\sin(4x) + C$$
$$= \boxed{\tfrac{1}{8}\,x\cos(4x) + (\tfrac{1}{4}x^2 - \tfrac{1}{32})\sin(4x) + C}$$

(e) $\int (4x+12)e^{x/3}\,\mathrm{d}x$

$u = 4x + 12$ and $\mathrm{d}v = e^{x/3}\,\mathrm{d}x$ gives $\mathrm{d}u = 4\,\mathrm{d}x$ and $v = 3e^{x/3}$.

$(4x+12)(3e^{x/3}) - \int 12e^{x/3}\,\mathrm{d}x = (12x+36)e^{x/3} - 36e^{x/3} + C = \boxed{12x\,e^{x/3} + C}$.

(f) $\int \cos(x)e^{2x}\,\mathrm{d}x$

$u = \cos(x)$ and $\mathrm{d}v = e^{2x}\,\mathrm{d}x$ gives $\mathrm{d}u = -\sin(x)\,\mathrm{d}x$ and $v = \frac{1}{2}e^{2x}$.

$$\int e^{2x}\cos(x)\,\mathrm{d}x = \tfrac{1}{2}\cos(x)e^{2x} - \int \tfrac{-1}{2}e^{2x}\sin(x)\,\mathrm{d}x$$
$$= \tfrac{1}{2}\cos(x)e^{2x} + \tfrac{1}{2}\int e^{2x}\sin(x)\,\mathrm{d}x.$$

We need parts again to do $\int e^{2x}\sin(x)\,\mathrm{d}x$.

New $u = \sin(x)$ and $\mathrm{d}v = e^{2x}$ (again) gives $\mathrm{d}u = \cos(x)\,\mathrm{d}x$ and $v = \frac{1}{2}e^{2x}$.

$$\int e^{2x}\sin(x)\,\mathrm{d}x = \tfrac{1}{2}\sin(x)e^{2x} - \int \tfrac{1}{2}e^{2x}\cos(x)\,\mathrm{d}x.$$

Therefore

$$\int e^{2x}\cos(x)\,\mathrm{d}x = \tfrac{1}{2}\cos(x)e^{2x} + \tfrac{1}{2}\left(\tfrac{1}{2}\sin(x)e^{2x} - \int \tfrac{1}{2}e^{2x}\cos(x)\,\mathrm{d}x\right)$$
$$\int \boldsymbol{e^{2x}\cos(x)\,\mathrm{d}x} = \tfrac{1}{2}\cos(x)e^{2x} + \tfrac{1}{4}\sin(x)e^{2x} - \tfrac{1}{4}\int \boldsymbol{e^{2x}\cos(x)\,\mathrm{d}x}$$
$$\tfrac{5}{4}\int \boldsymbol{e^{2x}\cos(x)\,\mathrm{d}x} = \tfrac{1}{2}\cos(x)e^{2x} + \tfrac{1}{4}\sin(x)e^{2x} + C$$
$$\int \boldsymbol{e^{2x}\cos(x)\,\mathrm{d}x} = \boxed{\tfrac{2}{5}\cos(x)e^{2x} + \tfrac{1}{5}\sin(x)e^{2x} + C}$$

225. Calculate the following definite integrals using integration by parts:

(a) $\int_0^6 (4x+12)e^{x/3}\,\mathrm{d}x$ We could use **Task 224(e)** and then

$$12x\,e^{x/3}\Big|_0^6 = 72\,e^2 - 0 = \boxed{72\,e^2}.$$

Alternatively, we can use the formula from the box, with $u = 4x+12$ and $v = 3e^{x/3}$:

$$\int_0^6 (4x+12)3e^{x/3}\,\mathrm{d}x + \int_0^6 3e^{x/3}4\,\mathrm{d}x = (4x+12)3e^{x/3}\Big|_{x=0}^{x=6}$$

$$\int_0^6 (4x+12)3e^{x/3}\,\mathrm{d}x + 36e^{x/3}\Big|_{x=0}^{x=6} = (4x+12)3e^{x/3}\Big|_{x=0}^{x=6}$$

$$\int_0^6 (4x+12)3e^{x/3}\,\mathrm{d}x + 36e^2 - 36 = 108e^2 - 36$$

$$\int_0^6 (4x+12)3e^{x/3}\,\mathrm{d}x = 108e^2 - 36e^2 = \boxed{72e^2}$$

(b) $\int_1^2 x\ln(x)\,\mathrm{d}x = \boxed{\ln(4) - \frac{3}{4}}$

(c) $\int_0^1 t\sin(\pi t)\,\mathrm{d}t = \boxed{\frac{1}{\pi}}$

(d) $\int_0^\pi x^4\cos(4x)\,\mathrm{d}x = \boxed{\frac{\pi}{8}}$

☆226. Prove that $\int_1^\pi \ln(x)\cos(x)\,\mathrm{d}x = \int_1^\pi \frac{-\sin(x)}{x}\,\mathrm{d}x$.

$$\int_1^\pi \ln(x)\cos(x)\,\mathrm{d}x + \int_1^\pi \frac{\sin(x)}{x}\,\mathrm{d}x = \ln(x)\sin(x)\Big|_{x=0}^{x=\pi} = 0$$

☆227. If $g(0) = 8$, $g(1) = 5$, and $\int_0^1 g(x)\,\mathrm{d}x = 2$, find the value of $\int_0^1 xg'(x)\,\mathrm{d}x$.

Parts with $u = x$ and $\mathrm{d}v = g'(x)\,\mathrm{d}x$ gives $\mathrm{d}u = \mathrm{d}x$ and $v = g(x)$, so as indefinite integrals,

$$\int xg'(x)\,\mathrm{d}x = \int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u = xg(x) - \int g(x)\,\mathrm{d}x.$$

As definite integrals,

$$\int_0^1 xg'(x)\,\mathrm{d}x = xg(x)\Big|_0^1 - \int_0^1 g(x)\,\mathrm{d}x = 1\cdot g(1) - 0\cdot g(0) - 2 = 5 - 2 = \boxed{3}.$$

228. Try each of the following methods to find $\int \sin(x)\cos(x)\,\mathrm{d}x$. (They are all possible.)

(a) Substitue $u = \sin(x)$, so $\mathrm{d}u = \cos(x)\,\mathrm{d}x$ and the integral is $\int u\,\mathrm{d}u$.

(b) Substitue $u = -\cos(x)$, so $du = \sin(x)\,dx$, and the integral is $\int -u\,du$.

(c) Substitute $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$, so the integral is $\frac{1}{2}\int \sin(2x)\,dx$.

(d) Do integration by parts with $u = \sin(x)$ and $dv = \cos(x)\,dx$.

(e) Do integration by parts with $u = \cos(x)$ and $dv = \sin(x)\,dx$.

☆(f) Compare your answers to parts (a) - (e).
See `https://youtu.be/-JR9-dgU7tU?t=520`

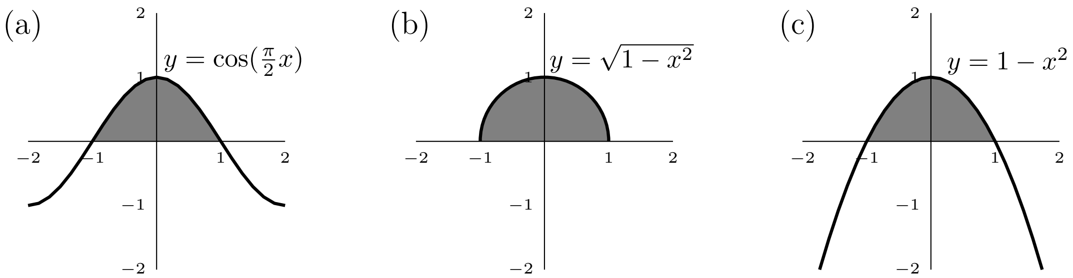
229. Find $\int 4x\cos(2-3x)\,dx$ and $\int (2-3x)\cos(4x)\,dx$.

(a) With $u = 4x$ and $dv = \cos(2-3x)\,dx$, we have $du = 4\,dx$ and $v = \frac{-1}{3}\sin(2-3x)$.

$$uv - \int v\,du = 4x(\tfrac{-1}{3}\sin(2-3x)) - \int 4(\tfrac{-1}{3}\sin(2-3x))\,dx$$
$$= \boxed{\tfrac{-4}{3}\sin(2-3x) + \tfrac{4}{9}\cos(2-3x) + C}.$$

(b) $\boxed{\dfrac{1}{2}\sin(4x) - \dfrac{3}{4}x\sin(4x) - \dfrac{3}{16}\cos(4x) + C}$

230. Give the area of each of the following shapes:



(a) $\int_{-1}^{1} \cos(\frac{\pi}{2}x)\,dx = \frac{2}{\pi}\sin(\frac{\pi}{2}x)\Big|_{x=-1}^{x=1} = \boxed{\dfrac{4}{\pi}} \approx 1.2732.$

(b) The area of a semicircle is $\frac{1}{2}\pi r^2 = \boxed{\dfrac{\pi}{2}} \approx 1.5708.$

(c) $\int_{-1}^{1} (1-x^2)\,dx = (x - \frac{1}{3}x^3)\Big|_{x=-1}^{x=1} = \boxed{\dfrac{4}{3}} \approx 1.3333.$

> The area between two curves of the form $y = f(x)$ is $\int_{\text{left}}^{\text{right}} \big(\text{top}(x) - \text{bottom}(x)\big)\,dx$.
>
> The area between two curves of the form $x = g(y)$ is $\int_{\text{bottom}}^{\text{top}} \big(\text{right}(y) - \text{left}(y)\big)\,dy$.

231. Find the area of the region bounded by $y = e^x, y = x + 5, x = -4$, and $x = 0$ (that is, the area between $y = e^x$ and $y = x + 5$ with $-4 \le x \le 0$).

$\int_{-4}^{0} \big((x+5) - e^x\big)\,dx = \boxed{11 + e^{-4}}$

232. What is the area of the region bounded by the curves $y = 20 - x^4$ and $y = 4$?

$$\int_{-2}^{2} \left((20 - x^4) - 4\right)\,dx = \boxed{\frac{256}{5}}$$

233. Find the area of the region bounded by the curves $x = y^2$ and $x = 1 + y - y^2$.

$$\int_{-1/2}^{1} \left((1 + y - y^2) - (y^2)\right)\,dy = \boxed{\frac{9}{8}}$$

234. Calculate the area of...

(a) the region bounded by the curves $y = x^2, y = 4x, x = 2, x = 3$.

$$\int_{2}^{3} \left(4x - x^2\right)\,dx = \boxed{\frac{11}{3}}$$

(b) the region bounded by the curves $y = x^2, y = 4x, y = 1, y = 4$.

$$\int_{1}^{4} \left(\sqrt{y} - \tfrac{1}{4}y\right)\,dy = \boxed{\frac{67}{24}}$$

(c) the region bounded by the curves $y = x^2$ and $y = 4x$.

$$\int_{0}^{4} \left(4x - x^2\right)\,dx = \int_{0}^{16} \left(\sqrt{y} - \tfrac{1}{4}y\right)\,dy = \boxed{\frac{32}{3}}$$

The volume of a solid can be calculated as

$$V = \int_{\text{left}}^{\text{right}} \big(\text{cross-section area}\big)\,dx = \int_{\text{bottom}}^{\text{top}} \big(\text{cross-section area}\big)\,dy.$$

For a "solid of revolution", the "disk method" uses

$$\pi \cdot (\text{radius})^2$$

as the cross-sectional area.

235. Find the volume of the solid formed by revolving (rotating) the region bounded by $y = 1 - x^2$ and $y = 0$ around the x-axis.

$$\int_{-1}^{1} \pi(1 - x^2)^2\,dx = \boxed{\frac{16}{15}\pi}$$

236. Find the volume of the solid formed by revolving the domain

$$\{(x, y) : x \geq 0, 2x \leq y \leq 6\}$$

around the y-axis.

$$\int_{0}^{6} \pi\left(\tfrac{y}{2}\right)^2\,dy = \boxed{18\pi}$$

For Winter 2023, you will not be asked about solids like the ones in Tasks 237 and 238.

☆237. For the solid formed by rotating the region from Task 234(c) around the x-axis,

(a) Set up an integral $\int \ldots\,dx$ for the volume using the washer method.

(b) and evaluate this integral. $\int_{0}^{4} \left(\pi(4x)^2 - \pi(x^2)^2\right)\,dx = \frac{2048}{15}\pi$

☆238. For the solid formed by rotating the region from Task 234(c) around the y-axis,

(a) set up an integral $\int \ldots \, dy$ for the volume using the washer method.

(b) and evaluate this integral. $\int_0^{16} \left(\pi(\sqrt{y})^2 - \pi(\tfrac{1}{4}y)^2\right) dy = \frac{128}{3}\pi$

239. Calculate each of the following integrals.
Some* require substitution, some** require parts, and some do not need either.

(a) $\int (x^4 + x^{1/2} + 4 + x^{-1})\, dx = \boxed{\tfrac{1}{5}x^5 + \tfrac{1}{3}x^{3/2} + \ln|x| + C}$

(b) $\int \left(x^2 + \sqrt{x} + \frac{\ln(81)}{\ln(3)} + \frac{1}{x}\right) dx = \boxed{\tfrac{1}{5}x^5 + \tfrac{1}{3}x^{3/2} + \ln|x| + C}$

(c) $\int (t + e^t)\, dt = \boxed{\frac{t^2}{2} + e^t + C}$

(d) $\int (t \cdot e^t)\, dt = \boxed{(t-1)e^t + C}$

(e) $\int (t^3 + e^{3t})\, dt = \boxed{\frac{t^4}{4} + \frac{e^{3t}}{3} + C}$

☆(f) $\int (t^3 \cdot e^{3t})\, dt = \boxed{\tfrac{1}{27}e^{3t}(9t^3 - 9t^2 + 6t - 2) + C}$

(g) $\int \frac{x}{x^2+1}\, dx = \boxed{\tfrac{1}{2}\ln(x^2+1) + C}$

(h) $\int \frac{x}{x^2-1}\, dx = \boxed{\tfrac{1}{2}\ln|x^2-1| + C}$

(i) $\int \frac{x^2-1}{x}\, dx = \boxed{\tfrac{1}{2}x^2 - \ln|x| + C}$

(j) $\int \frac{1}{x^2-1}\, dx = \boxed{\tfrac{1}{2}\ln|1-x| - \tfrac{1}{2}\ln|x+1| + C}$

☆(k) $\int \frac{1}{x^2+1}\, dx = \boxed{\arctan(x) + C}$

(ℓ) $\int \frac{y}{\sqrt{y^2+1}}\, dy = \boxed{\sqrt{y^2+1} + C}$

☆(m) $\int \frac{1}{\sqrt{y^2+1}}\, dy = \boxed{\ln(y + \sqrt{y^2+1}) + C}$

(n) $\int t\ln(t)\, dt = \boxed{\tfrac{1}{2}t^2\ln(t) - \tfrac{1}{4}t^4 + C}$

(o) $\int \frac{3t-12}{\sqrt{t^2-8t+6}}\, dt = \boxed{3\sqrt{t^2-8t+6} + C}$

(p) $\int \frac{1}{\sqrt{x-1}}\, dx = \boxed{2\sqrt{x-1} + C}$

(q) $\int \frac{x}{\sqrt{x-1}}\, dx = \boxed{\tfrac{2}{3}(x+2)\sqrt{x-1} + C}$

(r) $\int y^3 \, dy = \boxed{\frac{1}{4}y^4 + C}$

(s) $\int y(y+1)(y-1) \, dy = \boxed{\frac{1}{4}y^4 - \frac{1}{2}y^2 + C}$

(t) $\int x \sin(2x) \, dx = \boxed{\dfrac{1}{4}\sin(2x) - \dfrac{1}{2}x\cos(2x) + C}$

(u) $\int x^3 \sin(2x^4) \, dx = \boxed{\frac{-1}{8}\cos(2x^4) + C}$

☆ (v) $\int x^7 \sin(2x^4) \, dx$

This can be done using parts and substitution together!
If we substitute $w = 2x^4$ then $dw = 8x^3 \, dx$ and, using $x^7 = x^4 \cdot x^3$, we have

$$\int x^7 \sin(2x^4) \, dx = \int x^4 \cdot \sin(2x^4) \cdot x^3 \, dx$$
$$= \int (\tfrac{1}{2}w) \cdot \sin(w) \cdot \tfrac{1}{8} \, dw = \frac{1}{16}\int w \sin(w) \, dw.$$

Using integration by parts on $\int w \sin(w) \, dw$ is exactly like **Task 224(a)**, so we get $\int w \sin(w) \, dw = \sin(w) - w\cos(w) + C$ and

$$\tfrac{1}{16}\sin(w) - \tfrac{1}{16}w\cos(w) + C = \boxed{\tfrac{1}{16}\sin(2x^4) - \tfrac{1}{8}x^4\cos(2x^4) + C}.$$

☆ (w) $\int \sin(2x^4) \, dx$

This integral is literally impossible to write nicely. There cannot exist any formula for $F(x)$ using only $+ - \times \div$ and compositions of polynomials, trig, exponentials, and logs such that $F'(x) = \sin(2x^4)$.

(x) $\int e^{5x} \cos(e^{5x}) \, dx = \boxed{\frac{1}{5}\sin(e^{5x}) + C}$

☆ (y) $\int x^5 \cos(x) \, dx = \boxed{(x^5 - 20x^3 + 120x)\sin(x) + (5x^4 - 60x^2 + 120)\cos(x) + C}$

(z) $\int e^{8\ln(t)} \, dt = \int t^8 \, dt = \boxed{\frac{1}{9}t^9 + C}$

* g, h, m, o, p, q, u, x. ** d, f, ℓ, n, t, v, y.